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Characterization of quantum teleportation processes by nonlinear quantum channel and quantum mutual entropy

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Abstract

The quantum mutual entropy was introduced by one of the present authors in 1983 as a quantum extension of the Shannon mutual information. It has been used for several studies such as quantum information transmission in optical communication and quantum irreversible processes.

In this paper, a nonlinear channel for a quantum teleportation process is rigorously constructed and the quantum mutual entropy is applied to characterize the quantum teleportation processes of Bennett et al. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Recently, some physical models have been constructed in the topics named "quantum teleportation", "quantum cryptography" and "quantum computation", by which the difference between classical and quantum information are clearly revealed [1,2,4]. There exist significant problems in these topics, some of which have been extensively studied.

On the other hand, independently from these developments in the above fields, mathematical formulations of quantum entropy, relative entropy and mutual entropy have been already done in the field named "quantum probability" and "quantum information" coming from von Neumann's formulation. In particular, the quantum mutual entropy, a natural mathematical extension of Shannon's mutual entropy, was formulated in 1983 [7] by one of the present authors and applied to several studies on quantum capacity [9,12], irreversible processes and chaotic behavior [8]. Some of the works related to the above three topics by means of the entropy were reported in [3,6,13,14].

In this paper quantum teleportation processes [1] are rigorously studied. It is shown that such a process is generally expressed as a nonlinear channel, and the quantum mutual entropy can be used to characterize the quantum teleportation processes.

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2. Quantum mutual entropy

In this section, the von Neumann entropy, the quantum channel and the quantum mutual entropy are briefly reviewed [5,10].

2.1. Quantum entropy and quantum channel

The quantum entropy now becomes a fundamental tool to analyze quantum physical phenomena. Von Neumann entropy is defined for a density operator (state) ρ by

$$S(\rho) = -\mathrm{tr}\rho \log \rho. \tag{1}$$

Since the spectral set of ρ is discrete, we can write the spectral decomposition of ρ as

$$\rho = \sum_n \lambda_n P_n, \tag{2}$$

where λ_n is an eigenvalue of ρ and P_n is the projection from $\mathcal{H}$ onto the eigenspace associated with λ_n. Therefore, if every eigenvalue λ_n is not degenerate, then the dimension of the range of P_n is 1 (we denote this by $\dim P_n = 1$). If a certain eigenvalue, say λ_n, is degenerated, then P_n can be further decomposed into one-dimensional orthogonal projections, so that ρ is expressed as

$$\rho = \sum_n \lambda_n E_n \tag{3}$$

with one-dimensional projections $\{E_n\}$. We call this decomposition a Schatten decomposition of ρ.

In order to construct an information transmission, we have to prepare for at least two dynamical systems: an input system and an output system. Each system can be characterized by each state. Therefore it is important to know the relation between an input state and an output state in order to study information transmission. This relation is described by a channel describing the state change in the course of information transmission.

In the classical communication theory each state of input or output system is described by a probability distribution, so that a channel causes a change of this probability distribution.

On the other hand, in quantum communication theory, each state of input or output system should be described by a certain noncommutative state (quantum state) such as a density operator and a positive normalized linear functional.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the separable complex Hilbert spaces describing an input system and an output system, respectively. Let $B(\mathcal{H}_k)$ $(k = 1, 2)$ be the set of all bounded linear operators on $\mathcal{H}_k$. Then the set $\mathfrak{S}(\mathcal{H}_k)$ of all states (density operators) on the Hilbert spaces $\mathcal{H}_k$ is

$$\mathfrak{S}(\mathcal{H}_k) = \{\rho \in B(\mathcal{H}_k);\ \rho^* = \rho,\ \rho \geq 0,\ \mathrm{tr}\rho = 1\}.$$

A quantum channel sends an input quantum state to an output quantum state, that is, a mapping from $\mathfrak{S}(\mathcal{H}_1)$ to $\mathfrak{S}(\mathcal{H}_2)$.

2.2. Quantum mutual entropy

In communication theory, the mutual entropy $I(\rho; \Lambda^*)$ expresses how much information carried by an input state ρ is correctly transmitted to an output state $\Lambda^*\rho$ through a channel Λ^*. Therefore the quantum mutual entropy (information) for a quantum channel Λ^* should satisfy the following conditions:

(m.1) If Λ^* is an identity map, then $I(\rho; \Lambda^*) = S(\rho)$.

(m.2) $0 \leq I(\rho; \Lambda^*) \leq S(\rho)$.

(m.3) If ρ is a classical state (i.e., probability distribution or probability measure) and a channel Λ^* is sending a classical state to a classical state (i.e., classical channel), then $I(\rho; \Lambda^*)$ equals to the classical mutual entropy of Shannon.

In order to define such a quantum mutual entropy, we need the notion of compound state corresponding the joint probability of classical system.

A compound state $\sigma \in \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ describes the correlation between an input state $\rho \in \mathfrak{S}(\mathcal{H}_1)$ and its out putstate $\Lambda^*\rho \in \mathfrak{S}(\mathcal{H}_2)$, hence it should satisfy the following conditions: (i) $\operatorname{tr}\sigma(A \otimes I) = \operatorname{tr}\rho A$ for any $A \in B(\mathcal{H}_1)$. (ii) $\operatorname{tr}\sigma(I \otimes B) = \operatorname{tr}\Lambda^*\rho B$ for any $B \in B(\mathcal{H}_2)$. (iii) If the system is classical, then σ is identical to the joint probability.

The compound state satisfying conditions (i) – (iii) was constructed in [7,11] as

$$\sigma_E = \sum_k \lambda_k E_k \otimes \Lambda^* E_k, \tag{4}$$

where E indicates the Schatten decomposition $\{E_n\}$ because it is generally not unique. This construction comes from the following considerations: One-dimensional projection E_k associated with an initial state ρ can be considered as an elementary event composed of the state ρ. Therefore, in order to express a correlation between ρ and $\Lambda^*\rho$, the compound state should contain the relation among input constituents $\{E_k\}$ and output ones $\{\Lambda^* E_k\}$. This compound state plays a similar role as the joint probability in classical system.

Using the compound state σ_E the quantum mutual entropy $I(\rho; \Lambda^*)$ satisfying the above conditions (m.1) – (m.3) was defined as follows [7,11]:

$$I(\rho; \Lambda^*) = \sup_E \{S(\sigma_E, \sigma_0);\ E = \{E_n\} \text{ of } \rho\}, \tag{5}$$

where

$$\sigma_0 = \rho \otimes \Lambda^*\rho \quad \text{and} \quad S(\sigma_E, \sigma_0) = \operatorname{tr}\sigma_E(\log\sigma_E - \log\sigma_0)$$

(the quantum relative entropy [15]).

Here the supremum is taken over all Schatten decompositions of ρ unless every eigenvalue of ρ is not generated. The mutual entropy is the amount of information transmitted from ρ to $\Lambda^*\rho$ through a channel Λ^*. According to inequality (m.2), the quantum mutual entropy can be applied to many kinds of physical processes such as irreversible processes and quantum computation [11]. Other important properties of this mutual entropy $I(\rho; \Lambda^*)$ was reported in [5,7,8,10].

3. Quantum channel for quantum teleportation

The quantum teleportation was studied by Bennett et al. [1] (for simplicity BBCJPW) and its a bit general scheme is expressed in the following steps:

Step 0: Alice has an unknown quantum state $\rho^{(1)}$ (on Hilbert space $\mathcal{H}_1$) and she wants to teleport it to Bob.

Step 1: For this purpose, we need two other Hilbert spaces $\mathcal{H}_2$ and $\mathcal{H}_3$. $\mathcal{H}_2$ is attached to Alice and $\mathcal{H}_3$ is to Bob. Prearrange a state $\sigma_{23} \in \mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{H}_3)$ having a certain correlation between Alice and Bob.

Step 2: BBCJPW took an EPR singlet state for σ_{23}. For this EPR singlet state σ_{23}, Alice performs the joint measurement by means of the kth projection $F_k^{(12)}$ on a tensor product Hilbert space (system) $\mathcal{H}_1 \otimes \mathcal{H}_2$. (According to the effect of EPR pair, after Alice's measurement, Bob's EPR state becomes $\rho^{(3)}$.)

Step 3: Bob is informed of the projection which was used by Alice for the measurement $F_k^{(12)}$. This information is completely transmitted from Alice to Bob without disturbance.

Step 4: $\rho^{(1)}$ is reconstructed from $\rho^{(3)}$ (Step 2) by means of the information which Bob got from Alice in the above Step 3.

The above steps can be exhibited by a channel $\Lambda^{*[k]} : \mathfrak{T}(\mathcal{H}_1) \longrightarrow \mathfrak{T}(\mathcal{H}_1)$ constructed by the following four maps (channels):

(1) $$\gamma^* : \mathfrak{T}(\mathcal{H}_1) \longrightarrow \mathfrak{T}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3),\ \gamma^*(\rho^{(1)}) = \rho^{(1)} \otimes \sigma^{(23)} \quad \forall \rho^{(1)} \in \mathfrak{T}(\mathcal{H}_1). \tag{6}$$
This channel γ^* expresses a coupling of an initial state $\rho^{(1)}$ with the entangled state $\sigma^{(23)}$ (e.g., EPR singlet state).

(2) $\pi_k^* : \mathfrak{T}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3) \longrightarrow \mathfrak{T}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3)$ is a state change describing the nonclassical effect (e.g., EPR effect) of the teleportation determined by such as Alice's measurement.
Since $F_k^{(12)}$ is a projection (Alice's measurement) on $\mathcal{H}_1 \otimes \mathcal{H}_2$, that is, $F_k^{(12)} = (F_k^{(12)})^* = (F_k^{(12)})^2$, the map $\pi_k^* : \mathfrak{T}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3) \longrightarrow \mathfrak{T}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3)$ is given by

$$\pi_k^*(\rho^{(123)}) = \frac{(F_k^{(12)} \otimes I^{(3)})\rho^{(123)}(F_k^{(12)} \otimes I^{(3)})}{\mathrm{tr}_{123}(F_k^{(12)} \otimes I^{(3)})\rho^{(123)}(F_k^{(12)} \otimes I^{(3)})} \quad \forall \rho^{(123)} \in \mathfrak{T}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3). \tag{7}$$

(3) $a^* : \mathfrak{T}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3) \longrightarrow \mathfrak{T}(\mathcal{H}_3)$,

$$\rho^{(3)} = a^*(\rho^{(123)}) = \mathrm{tr}_{12}\,\rho^{(123)} \quad \forall \rho^{(123)} \in \mathfrak{T}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3). \tag{8}$$

Here tr_{12} is the partial trace: $\langle \Phi_1, \mathrm{tr}_{12}\, Q\Phi_2\rangle \equiv \sum_n \langle \Psi_n \otimes \Phi_1,\ Q\Psi_n \otimes \Phi_2\rangle$ for any $Q \in B(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3)$, any CONS $\{\Psi_n\}$ of $\mathcal{H}_1 \otimes \mathcal{H}_2$ and any $\Phi_1, \Phi_2 \in \mathcal{H}_3$.
This channel a^* represents a reduction from the state $\rho^{(123)}$ to Bob's state $\rho^{(3)}$ after Alice's measurement, where tr_{12} is the partial trace with respect to $\mathcal{H}_1 \otimes \mathcal{H}_2$.

(4) In the teleportation scheme by BBCJPW, there exists a unitary channel $u_k^* : \mathfrak{T}(\mathcal{H}_3) \longrightarrow \mathfrak{T}(\mathcal{H}_1)$,

$$u_k^*(\rho^{(3)}) = U_k \rho^{(3)} U_k^* \quad \forall \rho^{(3)} \in \mathfrak{T}(\mathcal{H}_3) \tag{9}$$

u_k^* is Bob's unitary operation corresponding to Alice's measurement $F_k^{(12)}$.
Therefore we obtain the channel $\Lambda^{*[k]} : \mathfrak{T}(\mathcal{H}_1) \longrightarrow \mathfrak{T}(\mathcal{H}_1)$

$$\Lambda^{*[k]} = u_k^* \circ \Lambda_{A\to B}^{*[k]} = u_k^* \circ a^* \circ \pi_k^* \circ \gamma^* \tag{10}$$

or equivalently

$$\Lambda^{*[k]}\rho^{(1)} = U_k[\mathrm{tr}_{12}\,\pi_k^*(\rho^{(1)} \otimes \sigma^{(23)})]U_k^* \quad \forall \rho^{(1)} \in \mathfrak{T}(\mathcal{H}_1), \tag{11}$$

where the subscript "[k]" means that the channels $\Lambda^{*[k]}$ and $\Lambda_{A\to B}^{*[k]}$ depends on the choice of Alice's measurement $F_k^{(12)}$.

Thus, the whole teleportation channel $\Lambda^{*[k]} : \mathfrak{T}(\mathcal{H}_1) \longrightarrow \mathfrak{T}(\mathcal{H}_1)$ is written as

$$\Lambda^{*[k]}\rho^{(1)} \equiv U_k \left[\mathrm{tr}_{12}\left[\frac{(F_k^{(12)} \otimes I^{(3)})(\rho^{(1)} \otimes \sigma^{(23)})(F_k^{(12)} \otimes I^{(3)})}{\mathrm{tr}_{123}(F_k^{(12)} \otimes I^{(3)})(\rho^{(1)} \otimes \sigma^{(23)})(F_k^{(12)} \otimes I^{(3)})}\right]\right] U_k^* \quad \forall \rho^{(1)} \in \mathfrak{T}(\mathcal{H}_1) \tag{12}$$

Note that our channel $\Lambda^{*[k]}$ is nonlinear.

4. Quantum mutual entropy for quantum teleportation

Let us consider the effect of the above nonlinear channel $\Lambda^{*[k]}_{A\to B}\rho^{(1)}$ before Bob's performance.

$$\Lambda^{*[k]}_{A\to B}\rho^{(1)} \equiv \mathrm{tr}_{12}\left[\frac{(F_k^{(12)}\otimes I^{(3)})(\rho^{(1)}\otimes\sigma^{(23)})(F_k^{(12)}\otimes I^{(3)})}{\mathrm{tr}_{123}(F_k^{(12)}\otimes I^{(3)})(\rho^{(1)}\otimes\sigma^{(23)})(F_k^{(12)}\otimes I^{(3)})}\right]. \tag{13}$$

Since the quantum mutual entropy $I(\rho;\Lambda^{*[k]})$ is invariant under a unitary channel and $\Lambda^{*[k]}\rho^{(1)} = u_k\circ\Lambda^{*[k]}_{A\to B}\rho^{(1)}$, we have

$$I(\rho^{(1)};\Lambda^{*[k]}) = I(\rho^{(1)};\Lambda^{*[k]}_{A\to B}). \tag{14}$$

We first consider $I(\rho^{(1)};\Lambda^{*[k]}_{A\to B})$ in the case that $\sigma^{(23)}$ is given by an EPR spin pair in a singlet state as BBCJPW scheme.

Let $\rho^{(1)}\in\Xi(\mathcal{H}_1)$ be an initial state which Alice wants to teleport and let its Schatten decomposition be

$$\rho^{(1)} = \sum_{j=1}^{2}\lambda_j|\phi_j\rangle\langle\phi_j|, \tag{15}$$

where

$$|\phi_j\rangle = a_j|\uparrow^{(1)}\rangle + b_j|\downarrow^{(1)}\rangle \text{ and } |a_j|^2+|b_j|^2 = 1 \qquad (j=1,2).$$

An entangled state $\sigma^{(23)}\in\Xi(\mathcal{H}_2\otimes\mathcal{H}_3)$ is assumed to be

$$\sigma^{(23)} = |\xi\rangle\langle\xi|, \tag{16}$$

where

$$|\xi\rangle = c|\uparrow^{(2)}\rangle\otimes|\downarrow^{(3)}\rangle + d|\downarrow^{(2)}\rangle\otimes|\uparrow^{(3)}\rangle \quad\text{and}\quad |c|^2+|d|^2 = 1.$$

If Alice's measurement $F_k^{(12)}$ is given by one of the Bell CONS$\{F_k^{(12)};k=1,\cdots,4\}$:

$$\begin{aligned} F_1^{(12)} &= |\Psi^{(-)}\rangle\langle\Psi^{(-)}|, & F_2^{(12)} &= |\Psi^{(+)}\rangle\langle\Psi^{(+)}|,\\ F_3^{(12)} &= |\Phi^{(-)}\rangle\langle\Phi^{(-)}|, & F_4^{(12)} &= |\Phi^{(+)}\rangle\langle\Phi^{(+)}| \end{aligned} \tag{17}$$

with

$$|\Psi^{(\pm)}\rangle = \sqrt{\frac{1}{2}}(|\uparrow^{(1)}\rangle\otimes|\downarrow^{(2)}\rangle \pm |\downarrow^{(1)}\rangle\otimes|\uparrow^{(2)}\rangle),$$

$$|\Phi^{(\pm)}\rangle = \sqrt{\frac{1}{2}}(|\uparrow^{(1)}\rangle\otimes|\uparrow^{(2)}\rangle \pm |\downarrow^{(1)}\rangle\otimes|\downarrow^{(2)}\rangle),$$

then Bob's state $\Lambda^{*[k]}_{A\to B}\rho^{(1)}\in\Xi(\mathcal{H}_3)$ becomes

$$\Lambda^{*[k]}_{A\to B}\rho^{(1)} = \sum_{j=1}^{2}c_j^{[k]}|\varphi_j^{[k]}\rangle\langle\varphi_j^{[k]}|, \tag{18}$$

where vectors $|\varphi_j^{[k]}\rangle$ and coefficients $c_j^{[k]}$ depend on the choice of Alice's measurement $F_k^{(12)}$ $(k = 1, 2, 3, 4)$. For example, in case of Alice's measurement $F_1^{(12)}$, a vector $|\varphi_j^{[1]}\rangle$ and a coefficient $c_j^{[1]}$ can be computed as follows:

$$|\varphi_j^{[1]}\rangle = \frac{da_j|\uparrow^{(3)}\rangle - cb_j|\downarrow^{(3)}\rangle}{\sqrt{|da_j|^2 + |cb_j|^2}}, \quad c_j^{[1]} = \frac{\lambda_j(|da_j|^2 + |cb_j|^2)}{\sum_{l=1}^{2}\lambda_l(|da_l|^2 + |cb_l|^2)}. \tag{19}$$

Other vectors $|\varphi_j^{[k]}\rangle$ and coefficients $c_j^{[k]}$ $(k = 2, 3, 4)$ are similarly computed.

In order to compute the mutual entropy, the compound state $\sum_{j=1}^{2}\lambda_j|\phi_j\rangle\langle\phi_j| \otimes \Lambda_{A\to B}^{*[k]}(|\phi_j\rangle\langle\phi_j|)$ is necessary. The state $\Lambda_{A\to B}^{*[k]}(|\phi_j\rangle\langle\phi_j|)$ is expressed by the above state $|\varphi_j^{[k]}\rangle$ as

$$\Lambda_{A\to B}^{*[k]}(|\phi_j\rangle\langle\phi_j|) = |\varphi_j^{[k]}\rangle\langle\varphi_j^{[k]}|. \tag{20}$$

Since $\{|\varphi_j^{[k])}\rangle\}$ is generally nonorthogonal, Schatten decomposition of $\Lambda_{A\to B}^{*[k]}\rho^{(1)}$ is written as

$$\Lambda_{A\to B}^{*[k]}\rho^{(1)} = \sum_{j=1}^{2} c_j^{[k]}\Lambda_{A\to B}^{*[k]}(|\phi_j\rangle\langle\phi_j|) = \sum_{j=1}^{2} c_j^{[k]}|\varphi_j^{[k]}\rangle\langle\varphi_j^{[k]}| = \sum_{k=1}^{2}\tilde{c}_j^{[k]}|\tilde{\varphi}_j^{[k]}\rangle\langle\tilde{\varphi}_j^{[k]}|. \tag{21}$$

Then we have the compound states σ_0 and σ_E for the computation of the quantum mutual entropy. Put $|\zeta_{jl}^{[k]}\rangle \equiv |\phi_j\rangle \otimes |\tilde{\varphi}_l^{[k]}\rangle$, $|\eta_j^{[k]}\rangle \equiv |\phi_j\rangle \otimes |\varphi_j^{[k]}\rangle$ with $\langle\zeta_{js}^{[k]}, \zeta_{lt}^{[k]}\rangle = \delta_{jl}\delta_{st}$ and $\langle\eta_j^{[k]}, \eta_l^{[k]}\rangle = \delta_{jl}$.

The quantum mutual entropy $I(\rho^{(1)}; \Lambda_{A\to B}^{*[k]})$ is computed as

$$I(\rho^{(1)}; \Lambda_{A\to B}^{*[k]}) = S(\sigma_E, \sigma_0) = -\sum_{l=1}^{2}\lambda_l\sum_{j=1}^{2}|\langle\tilde{\varphi}_j^{[k]}, \varphi_l^{[k]}\rangle|^2 \log\tilde{c}_j^{[k]}. \tag{22}$$

This is a general expression of the quantum mutual entropy for the quantum teleportation processes.

Now we apply the above formula to the BBCJPW scheme and rigorously examine it in the sense of the information conservation.

First we consider the original quantum teleportation by BBCJPW, in which the parameters c and d are set

$$c = \sqrt{\frac{1}{2}}, \qquad d = -\sqrt{\frac{1}{2}}. \tag{23}$$

Then we have $\tilde{c}_l^{[k]} = c_l^{[k]} = \lambda_l$ and $\langle\tilde{\varphi}_l^{[k]}, \varphi_j^{[k]}\rangle = \delta_{lj}$ $(l, j = 1, 2)$ from (19) and (21). Hence, we obtain the quantum mutual entropy:

$$I(\rho^{(1)}; \Lambda_{A\to B}^{*[k]}) = -\sum_{l=1}^{2}\lambda_l \log\lambda_l = S(\rho^{(1)}). \tag{24}$$

This result means the original quantum teleportation can perfectly send the initial information $S(\rho^{(1)})$ of Alice to Bob.

In [1], BBCJPW discussed more general case, namely, $\sigma^{(23)}$ is given by a pair of N-state particles in an entangled state. In this case, we almost similarly compute $I(\rho^{(1)}; \Lambda_{A\to B}^{*[nm]})$ by the followings steps:

Let $\rho^{(1)} \in \Xi(\mathcal{H}_1)$ be an initial state whose Schatten decomposition is given by

$$\rho^{(1)} = \sum_{j=0}^{N-1}\lambda_j|\phi_j\rangle\langle\phi_j|, \tag{25}$$

where

$$|\phi_j\rangle = \sum_{l=0}^{N-1} a_{jl}|l\rangle \left(\sum_{l=0}^{N-1} |a_{jl}|^2 = 1 (j = 0, 1, \ldots, N-1) \right) \text{ with CONS}\{|l\rangle\}.$$

The state vector $|\xi\rangle$ of the entangled state $\sigma_{23} = |\xi\rangle\langle\xi| \in \mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{H}_3)$ is assumed to be given by

$$|\xi\rangle = \sum_{j=0}^{N-1} w_j |j\rangle \otimes |j\rangle, \quad \sum_{j=0}^{N-1} |w_j|^2 = 1. \tag{26}$$

In this case, Alice's measurement is performed with a projection. In this case, Alice's measurement is performed with a projection

$$F_{nm}^{(12)} = |\psi_{nm}\rangle\langle\psi_{nm}|,$$

where

$$|\psi_{nm}\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \mathrm{e}^{\frac{2\pi i j n}{N}} |j\rangle \otimes |(j+m) \mod N\rangle.$$

Then Bob's state $\Lambda_{A\to B}^{*[nm]}\rho^{(1)} \in \mathfrak{S}(\mathcal{H}_3)$ becomes

$$\Lambda_{A\to B}^{*[nm]}\rho^{(1)} = \sum_{j=0}^{N-1} p_j^{[nm]} |d_j^{[nm]}\rangle\langle d_j^{[nm]}|, \tag{27}$$

where

$$|d_j^{[nm]}\rangle = \frac{\frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} \mathrm{e}^{\frac{2\pi i l n}{N}} a_{jl} w_{\{(l+m) \bmod N\}} |(l+m) \bmod N\rangle}{\sqrt{\sum_{s=0}^{N-1} |a_{js}|^2 |w_{\{(s+m) \bmod N\}}|^2}},$$

$$p_j^{[nm]} = \frac{\lambda_j \left(\sum_{l=0}^{N-1} |a_{jl}|^2 |w_{\{(s+m) \bmod N\}}|^2 \right)}{\sum_{t=0}^{N-1} \lambda_t \left(\sum_{s=0}^{N-1} |a_{ts}|^2 |w_{\{(s+m) \bmod N\}}|^2 \right)}. \tag{28}$$

Since $\{|d_j^{[nm]}\rangle\}$ is generally nonorthogonal, we have the following Schatten decomposition:

$$\Lambda_{A\to B}^{*[nm]}\rho^{(1)} = \sum_{j=0}^{N-1} \tilde{p}_j^{[nm]} |\tilde{d}_j^{[nm]}\rangle\langle\tilde{d}_j^{[nm]}|. \tag{29}$$

Then the quantum mutual entropy $I(\rho^{(1)}; \Lambda_{A\to B}^{*[nm]})$ is

$$I(\rho^{(1)}; \Lambda_{A\to B}^{*[nm]}) = -\sum_{l=0}^{N-1} \lambda_l \sum_{j=0}^{N-1} |\langle\tilde{d}_j^{[nm]}, d_l^{[nm]}\rangle|^2 \log \tilde{p}_j^{[nm]}. \tag{30}$$

In this general case, if we take $\omega_i = 1/\sqrt{N}$ $(i = 0, 1, \ldots, N-1)$, then our quantum mutual entropy is equivalent to $S(\rho^{(1)})$ as (24), which tells us that the original theory of BBCJPW is also rigorously proved in this case.

5. Conclusion

In this paper we constructed the general expression of the quantum teleportation channel based on the original teleportation scheme by Bennett et al. Moreover we rigorously derived the quantum mutual entropy for the information transmission in the quantum teleportation channel.

From these results we conclude that in the BBCJPW scheme the initial information can be perfectly transmitted through the linear quantum teleportation channel. In a general quantum teleportation the associated channel becomes nonlinear and the conservation of information is rather difficult to be studied, on which we are working now.

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